

Canonizing curvature squared action in the presence of lapse function

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Abstract

Lapse function appears as Lagrange multiplier in Einstein-Hilbert action and its variation leads to the (00) equation of Einstein, which corresponds to the Hamiltonian constraint equation. In higher order theory of gravity the situation is not that simple. Here, we take up curvature squared action being supplemented by an appropriate boundary term in the background of Robertson-Walker minisuperspace metric, and show how to identify the constraint equation and formulate the Hamiltonian without detailed constraint analysis. The action is finally expressed in the canonical form, where, the lapse function appears as Lagrange multiplier. Canonical quantization yields Schrödinger like equation, with excellent features.

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1 Introduction

Explaining the cosmic evolution, taking only geometric terms in the action, has turned out to be an important issue presently, since we do not have a scalar field at hand, as yet. Particularly, an action in the form, $A = \int [\alpha R + \beta R^2 + \gamma R^{-1}] \sqrt{-g} d^4x$, apparently can challenge scalar field theories. The dominance of R^{-1} term at the very late stage of evolution leads to effective negative pressure, sufficient to explain the SnIa data with an accelerating phase, the dominance of R term in the middle, keeps the nucleosynthesis and the growth of perturbation necessary for structure formation, unchanged from Friedmann model, while the dominance of R^2 term in the early universe leads to inflation without invoking phase transition [1], [2]. Modification of the Einstein-Hilbert action by including curvature squared terms ($R^2, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda}$), $C_{\mu\nu\rho\lambda}$ being the Weyl tensor, is also important in many other respect. It leads to a renormalizable theory of gravity [3], even while interacting with matter [4] and is also asymptotically free [5]. Unitarity of higher derivative quantum theory of gravity has also been established [6]. Further, the Euclidean form of the Einstein-Hilbert action is not positive definite and therefore the functional integral corresponding to the ground state wave function of the universe diverges badly. A positive definite action that includes R^2 -term, in the form [7], [8]

$$S = -\frac{1}{4} \int d^4X \sqrt{-g} [AC_{ijkl}^2 + B(R - 4\Lambda)^2],$$

leads to a convergent integral for the ground state wave function, and it reduces to Einstein-Hilbert action in the weak field limit. Additionally, canonical quantization of the above action leads to a Schrödinger like equation, where an internal variable acts as the time parameter [7], [9], [10], [11]. A string inspired theory of gravity [12] and the 4-dimensional Brane world effective action [13] also contain such terms. In view of the above discussion, it turns out to be an important issue to include curvature squared term in the gravitational action and to study its quantum cosmological consequence, since it plays a dominating role only in the early universe. This requires canonical formulation of the theory.

Hamiltonian formulation of a gravitational action containing curvature squared term is non trivial. Gibbons and

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Hawking [14] have argued that the Einstein-Hilbert gravitational action is incomplete, unless it is supplemented by an appropriate surface term that involves the integral over the boundary of space-time of the trace of the second fundamental form. To get a unique Hamiltonian of a theory, a unique boundary term is required which must be expressed as the integral over the boundary of space-time of some quantity. Without an appropriate boundary term the Hamiltonian and the corresponding quantum description of a theory usually turns out to be completely wrong [15]. The problem regarding canonical quantization of any other higher order theory of gravity is that, they are associated with different boundary terms [16] corresponding to different variational techniques, and apparently none of them is a total derivative term. Now, canonical formulation of any higher order theory of gravity also requires the introduction of an auxiliary variable. Sanyal [15] has shown that curvature squared action should be supplemented by a unique boundary term in the form, $\Sigma = 4\beta \int ({}^4R)K\sqrt{h} d^3x$, where, K is the trace of the extrinsic curvature, h is the determinant of the metric of the three space and β is the coupling constant. Apparently, such a term is not a total derivative term. But then, $\Sigma = \sigma_1 + \sigma_2$, where both can be individually expressed as total derivative terms. The first one, $\sigma_1 = 4\beta \int ({}^3R)K\sqrt{h} d^3x$ gets cancelled before the introduction of auxiliary variable and the other, viz., $\sigma_2 = 4\beta \int ({}^4R - {}^3R)K\sqrt{h} d^3x$ gets cancelled after the introduction of the auxiliary variable. This makes R^2 theory of gravity apparently as complete as Einstein-Hilbert action, when it is expressed as

$$A = \beta \int R^2 \sqrt{-g} d^4x + \sigma_1 + \sigma_2. \quad (1)$$

Although, Σ has been found in view of Robertson-Walker minisuperspace metric, however it has also been tested successfully in a few other anisotropic minisuperspace metrics, viz., Kantowski-Sachs, axially symmetric Bianchi-I and Bianchi - III [15]. It is also important to note that the standard variational principle (metric formalism) of $F(R)$ theory of gravity yields a surface term [17] in the form $2 \int_{\Sigma} \sqrt{h} F_{,R} K d^3x$, which reduces to the above surface term Σ , for $F(R) = R^2$ and $\beta = 1$. The surface term for $F(R)$ theory of gravity also reduces to the Gibbons-Hawking term, when $F(R) = \frac{R}{16\pi G}$. As already mentioned, Σ does not appear to be a total derivative term. What we have achieved in our earlier work [15], is to show that expressing $\Sigma = \sigma_1 + \sigma_2$ and under suitable choice of auxiliary variable, the terms individually appear as total derivative terms.

Still, there remains an unsolved important issue, which is presently our concern. The space-time metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, under $3+1$ decomposition can be expressed as,

$$ds^2 = - (N^2 - N_i N^i) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j, \quad (2)$$

where, N and N_i are the lapse function and shift vector respectively. In view of such decomposition, Einstein-Hilbert action when supplemented by Gibbons-Hawking surface term,

$$A = \int \frac{1}{16\pi G} (R - 2\Lambda) \sqrt{-g} d^4x + \frac{1}{8\pi G} \int \sqrt{h} K d^3x, \quad (3)$$

leads to a canonical action in terms of the basic variables h_{ij} and its canonical conjugate momenta π_{ij} , in the form,

$$A = \int [\dot{h}_{ij} \pi^{ij} - H_c - H_{ci}] d^3x dt = \int [\dot{h}_{ij} \pi^{ij} - N\mathcal{H} - N^i \mathcal{H}_i] d^3x dt, \quad (4)$$

where, the lapse function N and the shift vector N^i appear as Lagrange multipliers. Variation of the action (4) with respect to the shift vector gives the super momentum constraint,

$$\mathcal{H}_i = 2D_j \pi_i^j = 0, \quad (5)$$

and variation with respect to the lapse function gives the super Hamiltonian constraint,

$$\mathcal{H} = (16\pi G) G_{ijkl} \pi_{ij} \pi^{kl} - \frac{1}{16\pi G} \sqrt{h} ({}^3R - 2\Lambda) = 0, \quad (6)$$

where, the metric on the superspace is expressed as,

$$G_{ijkl} = \frac{1}{2\sqrt{h}} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}). \quad (7)$$

The canonical quantization is then straight forward, which gives the so called Wheeler-deWitt equation. Thus to claim that the action (1) is as complete as Einstein-Hilbert action (3), it should also be expressed in the canonical form (4). Such a construction has turned out to be impossible taking into account the whole superspace, and it has not so far been tried in minisuperspace model too. In the following section, we attempt this issue in the Robertson-Walker minisuperspace model, which accommodates lapse function only. In section 3, we extend our work to include Einstein-Hilbert action in addition to curvature squared term.

2 Canonical formulation of (scalar) curvature squared action

A general scale invariant action is expressed as,

$$A = \int [\alpha C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda} + \beta R^2] \sqrt{-g} d^4x.$$

Since the Weyl tensor vanishes in the Robertson-Walker minisuperspace metric,

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \quad (8)$$

hence the above action, in the presence of a cosmological constant Λ and supplemented by appropriate boundary term, reduces to

$$A = \beta \int \sqrt{-g} d^4x \left[R^2 - \frac{2\Lambda}{\beta} \right] + \sigma_1 + \sigma_2, \quad (9)$$

The Ricci scalar, $R = \frac{6}{N^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + N^2 \frac{k}{a^2} - \frac{\dot{N}\dot{a}}{Na} \right)$, under the choice $h_{ij} = a^2 = z$ takes the form,

$$R = \frac{6}{N^2} \left[\frac{\ddot{z}}{2z} + N^2 \frac{k}{z} - \frac{\dot{z}\dot{N}}{2zN} \right], \quad (10)$$

and the above action now reads,

$$A = c\beta \int \left[\frac{9\ddot{z}^2}{N^3\sqrt{z}} + 36k \left(\frac{\ddot{z}}{N\sqrt{z}} - \frac{\dot{N}\dot{z}}{N^2\sqrt{z}} \right) - \frac{18\dot{N}\dot{z}\ddot{z}}{N^4\sqrt{z}} + \frac{9\dot{N}^2\dot{z}^2}{N^5\sqrt{z}} + \frac{36Nk^2}{\sqrt{z}} - \frac{2N\Lambda z^{\frac{3}{2}}}{\beta} \right] dt + \sigma_1 + \sigma_2, \quad (11)$$

where, as already mentioned,

$$\Sigma = \sigma_1 + \sigma_2 = 4\beta \int \sqrt{h} K^3 R d^3x + 4\beta \int \sqrt{h} K (^4R - ^3R) d^3x = 4\beta \int \sqrt{h} K^4 R d^3x. \quad (12)$$

and the constant c is the volume of the three space. Under integration by parts the first bracketed terms in the above action yield a counter term, that gets cancelled with σ_1 and we are left with,

$$A = B \int \left[\frac{9\ddot{z}^2}{N^3\sqrt{z}} - \frac{18\dot{N}\dot{z}\ddot{z}}{N^4\sqrt{z}} + \frac{9\dot{N}^2\dot{z}^2}{N^5\sqrt{z}} + \frac{18k\dot{z}^2}{Nz^{\frac{3}{2}}} + \frac{36Nk^2}{\sqrt{z}} - \frac{2N\Lambda z^{\frac{3}{2}}}{\beta} \right] dt + \sigma_2. \quad (13)$$

where, we have chosen $B = c\beta$. At this stage we introduce the auxiliary variable,

$$Q = \frac{\partial A}{\partial \ddot{z}} = 18B \left[\frac{\ddot{z}}{N^3\sqrt{z}} - \frac{\dot{N}\dot{z}}{N^4\sqrt{z}} \right], \quad (14)$$

and express the action in the canonical form as,

$$A = B \int \left[\frac{Q\ddot{z}}{B} - \frac{\dot{N}\dot{z}}{BN} Q - \frac{N^3\sqrt{z}}{36B^2} Q^2 + \frac{18k\dot{z}^2}{Nz^{\frac{3}{2}}} + \frac{36Nk^2}{\sqrt{z}} - \frac{2N\Lambda z^{\frac{3}{2}}}{\beta} \right] dt + \sigma_2. \quad (15)$$

Now the first term is integrated by parts and the total derivative term gets cancelled with σ_2 , and we are finally left with,

$$A = \int \left[-\dot{Q}\dot{z} - \frac{\dot{N}}{N}\dot{z}Q + \frac{18Bk\dot{z}^2}{Nz^{\frac{3}{2}}} - \frac{N^3\sqrt{z}}{36B} Q^2 + B \left(\frac{36Nk^2}{\sqrt{z}} - \frac{2\Lambda}{\beta} Nz^{\frac{3}{2}} \right) \right] dt. \quad (16)$$

The canonical momenta are,

$$p_z = -\dot{Q} - \frac{\dot{N}}{N}Q + 36Bk \frac{\dot{z}}{Nz^{\frac{3}{2}}}, \quad p_Q = -\dot{z}, \quad p_N = -\dot{z} \frac{Q}{N}. \quad (17)$$

The Q variation equation gives back the definition of Q given in (14). The z variation equation is,

$$\ddot{Q} + \frac{\dot{N}}{N}Q + \frac{\dot{N}}{N}\dot{Q} - \frac{\dot{N}^2}{N^2}Q - 36Bk \left(\frac{\ddot{z}}{Nz^{\frac{3}{2}}} - \frac{\dot{N}\dot{z}}{N^2z^{\frac{3}{2}}} - \frac{3\dot{z}^2}{4Nz^{\frac{5}{2}}} \right) - \frac{N^3Q^2}{72B\sqrt{z}} - 18Bk^2\frac{N}{z^{\frac{3}{2}}} + \frac{3B}{\beta}N\Lambda\sqrt{z} = 0, \quad (18)$$

and the N variation equation is

$$-\frac{\ddot{z}Q}{N} - \frac{\dot{z}\dot{Q}}{N} + 18Bk\frac{\dot{z}^2}{N^2z^{\frac{3}{2}}} + \frac{N^2\sqrt{z}}{12B}Q^2 - B \left(\frac{36k^2}{\sqrt{z}} - \frac{2\Lambda}{\beta}z^{\frac{3}{2}} \right) = 0. \quad (19)$$

In view of the definition of momenta given in (17), it is clear that neither Q nor N is invertible, which signals the presence of a constraint in the theory. This is also apparent from the fact that the Hessian determinant vanishes, i.e., $\mathbf{H} = |\sum_{i,j} \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}| = 0$. Apparently, Q and N are having the same status in the action. Nevertheless, we know that Q is only an auxiliary variable and it has been introduced in the action keeping its canonical form intact. As a result Q variation equation gives back the definition (14) of Q and must not give any dynamics. Now, the third signal for the presence of constraint is that a particular variable is non-dynamical, i.e., none of the field equations contain second derivative of that variable. But then, we observe that second derivative of N appears in equation (18). This is definitely confusing. Finally, the fourth signal for the presence of a constraint is that one of the field equations must not be dynamical, i.e., it must not contain second derivative term. However, both the equations (18) and (19) contain second derivative terms. Thus, the action (16) contains a constraint, but the constraint equation is hidden. The reason is, unlike the case of Einstein-Hilbert action, the action (16) contains first derivative of N . In fact, one can choose, a variable $Q' = NQ$ to get rid of \dot{N} term from the action (16). In the process, N acts as Lagrange multiplier and \dot{N} term disappears from the z variation equation (18) while, \ddot{z} term disappears from the N variation equation (19). As a result, equation (19) being free from second derivative term, stands as the constraint equation. Instead, one can also remove \ddot{z} term from equation (19) in view of the definition of Q given in (14). Thus equation (19) takes the following form, viz.,

$$-\frac{\dot{z}\dot{Q}}{N} - \frac{\dot{N}\dot{z}Q}{N^2} + 18Bk\frac{\dot{z}^2}{N^2z^{\frac{3}{2}}} + \frac{N^2\sqrt{z}}{36B}Q^2 - B \left(\frac{36k^2}{\sqrt{z}} - \frac{2\Lambda}{\beta}z^{\frac{3}{2}} \right) = 0. \quad (20)$$

This is the equation we were in search of, which does not contain second derivative term and hence is a constraint of the system under consideration. It can be easily verified that this is the Hamiltonian of the system in disguise,

$$H_c = N \left[-\frac{\dot{z}\dot{Q}}{N} - \frac{\dot{N}\dot{z}Q}{N^2} + 18Bk\frac{\dot{z}^2}{N^2z^{\frac{3}{2}}} + \frac{N^2\sqrt{z}}{36B}Q^2 - B \left(\frac{36k^2}{\sqrt{z}} - \frac{2\Lambda}{\beta}z^{\frac{3}{2}} \right) \right], \quad (21)$$

which is constrained to vanish in view of equation (20). So, as in the case of Einstein-Hilbert action, here too the Hamiltonian can be obtained under the variation of the lapse function, but then one has to utilize the definition of auxiliary variable in addition. Now, the next question is, how to express the Hamiltonian in terms of the phase-space variables? This usually requires detailed constraint analysis [18], where, the constraint in the configuration space variable is equation (20) and that in phase space variable is $Qp_Q - Np_N = 0$, as is observed in view of the canonical momenta (17). However, we show that even without going into the details of constraint analysis, the Hamiltonian in terms of the phase space variables may be obtained in a straight forward manner. This is possible because N acts only as a Lagrange multiplier. The definitions of canonical momenta (17) yield,

$$p_Q p_z = \dot{z}\dot{Q} + \frac{\dot{N}}{N}\dot{z}Q - 36Bk\frac{\dot{z}^2}{Nz^{\frac{3}{2}}}.$$

So,

$$-\dot{z}\dot{Q} - \frac{\dot{N}\dot{z}Q}{N} + 18Bk\frac{\dot{z}^2}{Nz^{\frac{3}{2}}} = -p_Q p_z - 18Bk\frac{\dot{z}^2}{Nz^{\frac{3}{2}}} = -p_Q p_z - \frac{18Bk}{Nz^{\frac{3}{2}}}p_Q^2,$$

where, we have replaced \dot{z} by p_Q , instead of p_N , since N acts as Lagrange multiplier and so the Hamiltonian must not contain p_N . Thus the Hamiltonian constraint equation in terms of the phase space variables is obtained as,

$$H_c = -p_Q p_z - \frac{18Bk}{Nz^{\frac{3}{2}}}p_Q^2 + \frac{N^3\sqrt{z}}{36B}Q^2 - BN \left(\frac{36k^2}{\sqrt{z}} - \frac{2\Lambda}{\beta}z^{\frac{3}{2}} \right) = 0. \quad (22)$$

But still this is not the end of the story. To express the curvature squared action in the canonical form (4) as in the case of Einstein-Hilbert action, we need to express H_c as $H_c = N\mathcal{H}$. But, first of all it is required to express the Hamiltonian in terms of the basic (configuration space) variables (instead of auxiliary variable) spanned by, $\{h_{ij}, \pi_{ij}, K_{ij}, \Pi_{ij}\}$ which essentially are $\{z, p_z, \dot{z}, p_{\dot{z}}\}$, in the Robertson-Walker minisuperspace metric. For this purpose and to avoid confusion, the standard choice is $\dot{z} = x$ (see [7], [9], [10], [11], [15]). Thus, our extended phase space is spanned by $\{z, p_z, x, p_x\}$. But here, to express $H_c = N\mathcal{H}$, so that \mathcal{H} remains free from N , we choose,

$$x = \frac{\dot{z}}{N}.$$

Hence,

$$Q = \frac{\partial A}{\partial \dot{z}} = \frac{\partial A}{\partial \dot{x}} \frac{d\dot{x}}{d\dot{z}} = \frac{p_x}{N}, \text{ and, } p_Q = -\dot{z} = -Nx,$$

So we need to replace Q by $\frac{p_x}{N}$ and p_Q by $-Nx$ in the above Hamiltonian. Thus, finally we are able to write,

$$H_c = N \left(xp_z + \frac{\sqrt{z}}{36B} p_x^2 - \frac{18Bk}{z^{\frac{3}{2}}} x^2 - 36B \frac{k^2}{\sqrt{z}} + \frac{2B\Lambda}{\beta} z^{\frac{3}{2}} \right) = N\mathcal{H} = 0. \quad (23)$$

It is now straight forward to express the action (15) as,

$$A = \int (\dot{z}p_z + \dot{x}p_x - N\mathcal{H}) dt \, d^3x, \quad (24)$$

and we have achieved our goal of expressing the curvature squared action in the canonical form with respect to the basic variables. We can also anticipate a general form of the above canonical action. Remember, $x = \frac{\dot{z}}{N} = 2\frac{\dot{\alpha}}{N} = -2K_{ij}$, and so $\dot{x} = -2\dot{K}_{ij}$, where, K_{ij} is the extrinsic curvature tensor. Again, $p_x = -\frac{1}{2}\Pi_{ij}$, Π_{ij} being the momentum canonically conjugate to K_{ij} . Hence we may write the above form of canonical action as,

$$A = \int (\dot{h}_{ij}\pi^{ij} + \dot{K}_{ij}\Pi^{ij} - N\mathcal{H}) dt \, d^3x, \quad (25)$$

where, π^{ij} is the momenta canonical to h_{ij} . This result that curvature squared action can be expressed in a general canonical form is of course new and exiting. Now, the canonical quantization of the Hamiltonian constraint equation is straight forward, which yields,

$$\frac{i\hbar}{\sqrt{z}} \frac{\partial \Psi}{\partial z} = -\frac{\hbar^2}{36Bx} \left(\frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi - 18Bk \left(\frac{x}{z^2} + \frac{2k}{zx} \right) \Psi + \frac{2B\Lambda}{\beta} \frac{z}{x} \Psi. \quad (26)$$

Again under a further change of variable, the above equation takes the look of the Schrödinger equation, viz.,

$$i\hbar \frac{\partial \Psi}{\partial \alpha} = -\frac{\hbar^2}{54B} \left(\frac{1}{x} \frac{\partial^2}{\partial x^2} + \frac{n}{x^2} \frac{\partial}{\partial x} \right) \Psi - B \left[\frac{12kx}{\alpha^{\frac{4}{3}}} + \frac{24k^2}{x\alpha^{\frac{2}{3}}} - \frac{4\Lambda}{3\beta x} \alpha^{\frac{2}{3}} \right] \Psi = \hat{H}_e \Psi \quad (27)$$

where, n is the operator ordering index and $\alpha = z^{\frac{3}{2}} = a^3$. Hence, the proper volume plays the role of internal time parameter. Note that the effective Hamiltonian

$$\hat{H}_e(x, \alpha) = -\frac{\hbar^2}{54B} \left(\frac{1}{x} \frac{\partial^2}{\partial x^2} + \frac{n}{x^2} \frac{\partial}{\partial x} \right) + V_e(x, \alpha), \quad (28)$$

is hermitian, where the effective potential V_e , given by,

$$V_e(x, \alpha) = -B \left[\frac{12kx}{\alpha^{\frac{4}{3}}} + \frac{24k^2}{x\alpha^{\frac{2}{3}}} - \frac{4\Lambda}{3\beta x} \alpha^{\frac{2}{3}} \right], \quad (29)$$

is a function of both the so called time variable α and x . The hermiticity of the effective Hamiltonian allows one to write the continuity equation for $n = -1$, as,

$$\frac{\partial \rho}{\partial \alpha} + \nabla \cdot \mathbf{J} = 0,$$

where, $\rho = \Psi^* \Psi$ and $\mathbf{J} = (\mathbf{J}_x, 0, 0)$ are the probability density and the current density respectively, with, $\mathbf{J}_x = \frac{i\hbar}{36Bx}(\Psi\Psi_{,x}^* - \Psi^*\Psi_{,x})$. It is important to note that the continuity equation in the above standard form is found only under the choice of the factor ordering index $n = -1$. Thus, factor ordering index has been fixed from physical argument. Further, there exists a continuous one parameter unitary group $U(\alpha)$, which may be expressed by the Dyson series, Viz.,

$$U(\alpha) = \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_{\alpha_0}^{\alpha} H_e(x, \alpha) d\alpha \right],$$

where, \mathcal{T} is the Dyson time ordering symbol. Finally, in the very early Universe when α is of the order of Planck's dimension, the term containing the cosmological constant remains subdominant in the effective potential and may be neglected. The extremization of the V_e then yields,

$$a = \sqrt{\frac{k}{2}}(t - t_0), \quad (30)$$

which has been obtained earlier [15]. This clearly depicts that power law inflation is an artefact of curvature squared action, and cosmological evolution must have started with a positive curvature parameter, $k > 0$. In the later epoch, Λ term starts playing a dominant role, but it is not possible to obtain a solution of the extremum of the potential in closed form, keeping the lambda term.

3 Einstein-Hilbert action being modified by curvature squared term

A general coordinate invariant fourth order action is expressed as,

$$A = \int \left[\frac{R - 2\Lambda}{16\pi G} + \beta R^2 + \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right] \sqrt{-g} d^4x. \quad (31)$$

This action can formally be cast to a unitary renomalizable quantum theory of gravity with positive energy states under Lee-Wick [5] prescription. In the Robertson-Walker minisuperspace metric the Weyl tensor vanishes and after being supplemented by appropriate boundary term, it is expressed as,

$$A = \int \left[\frac{R - 2\Lambda}{16\pi G} + \beta R^2 \right] \sqrt{-g} d^4x + \sigma + \sigma_1 + \sigma_2, \quad (32)$$

where, $\sigma = \frac{1}{8\pi G} \int \sqrt{h} K d^3x$ is the Gibbons-Hawking boundary term. This action leads to inflation without phase transition, followed by reheating [2]. Hence, if one is interested in canonical formulation of a general coordinate invariant fourth order action in Robertson-Walker minisuperspace metric, it is sufficient to start with action (32). It is important to note that action (32) may be cast into a positive definite one under appropriate choice of β , and has a newtonian gravity long-distance limit as a classical theory. However, a more general quadratic action is expressed as [19],

$$A = \int \left[\frac{R - 2\Lambda}{16\pi G} + \alpha R^2 + \gamma R_{\mu\nu} R^{\mu\nu} + \epsilon R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \lambda \square R \right] \sqrt{-g} d^4x. \quad (33)$$

Instead of taking Kretschman scalar squared term, it is customary to express it as $\epsilon\chi$, where,

$$\chi = \frac{1}{32\pi^2} \int (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2) \sqrt{-g} d^4x.$$

χ , also known as Gauss-Bonnet term is topologically invariant and so its functional derivative vanishes in four dimension. Further, $\square R$ is manifestly covariant total divergent term and is usually ignored. Hence, the above action reduces to,

$$A = \int \left[\frac{R - 2\Lambda}{16\pi G} + \alpha R^2 + \gamma R_{\mu\nu} R^{\mu\nu} \right] \sqrt{-g} d^4x, \quad (34)$$

which is also the action for the gravitational sector under BRST symmetry [20]. The above action may also be recast as,

$$A = \int \left[\frac{R - 2\Lambda}{16\pi G} + \beta R^2 + \gamma (R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) \right] \sqrt{-g} d^4x, \quad (35)$$

just by choosing, $\beta = \alpha + \frac{1}{3}\gamma$. In the Robertson-Walker minisuperspace model, $\int (R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)\sqrt{-g}d^4x$ is again a total derivative term and thus is ignored. Hence, the above action (35) again reduces to the one given in (32), in the Robertson-Walker metric. It is interesting to note that the Weyl tensor squared term,

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$

may also be expressed as,

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2) + 2(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2),$$

where, the first term is the Gauss-Bonnet term. Therefore, under the assumption of homogeneity and isotropy of space time, dubbed as the cosmological principle, both the actions (31) and (33) or (34) in disguise, reduce to (32), which we take up for the present purpose. In fact, instead of expressing (34) in the form given in (35), one could also express it as,

$$A = \int \left[\frac{R - 2\Lambda}{16\pi G} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2) \right] \sqrt{-g}d^4x,$$

which reduces to

$$A = \int \left[\frac{R - 2\Lambda}{16\pi G} + \beta R_{\mu\nu}R^{\mu\nu} \right] \sqrt{-g}d^4x,$$

in the Robertson-Walker minisuperspace model under consideration. This action is more interesting than (32), since in the gauge theoretic formulation of gravitation, R^2 only gives a scalar mode, while $R_{\mu\nu}R^{\mu\nu}$ contains vector mode in the form of massless spin 2 gravitons. However, at present we pay our attention to action (32), which, as already mentioned, can be expressed as a positive definite action under appropriate choice of β , and has a newtonian gravity long-distance limit as a classical theory.

Following the same procedure as in section 2, one can express action (32) in the following canonical form,

$$A = \int \left[\frac{c}{16\pi G} \left(-\frac{3\dot{z}^2}{2N\sqrt{z}} + 6kN\sqrt{z} - 2\Lambda Nz^{\frac{3}{2}} \right) - Q\dot{z} - \frac{\dot{N}z}{N}Q - \frac{N^3\sqrt{z}}{36B}Q^2 + \frac{18Bk\dot{z}^2}{Nz^{\frac{3}{2}}} + \frac{36BNk^2}{\sqrt{z}} \right] dt, \quad (36)$$

where, the auxiliary variable Q and the canonical momenta, p_Q and p_N have the same expressions as found in equations (14) and (17) of the previous section, while p_z is different, viz.,

$$p_z = -\frac{3c}{16\pi G} \left(\frac{\dot{z}}{N\sqrt{z}} \right) - \dot{Q} - \frac{\dot{N}}{N}Q + \frac{36Bk\dot{z}}{Nz^{\frac{3}{2}}}. \quad (37)$$

The N variation equation now is,

$$-\frac{c}{16\pi G} \left(\frac{3\dot{z}^2}{2N^2\sqrt{z}} + 6k\sqrt{z} - 2\Lambda z^{\frac{3}{2}} \right) - \frac{\ddot{z}}{N} - \frac{\dot{z}\dot{Q}}{N} + 18Bk\frac{\dot{z}^2}{N^2z^{\frac{3}{2}}} + \frac{N^2\sqrt{z}}{12B}Q^2 - \frac{36Bk^2}{\sqrt{z}} = 0, \quad (38)$$

which is supposed to be the Hamilton constraint equation. As before, removing \ddot{z} term, in view of the definition of the auxiliary variable (14), one can easily verify that this is again the Hamiltonian of the system in disguise,

$$H_c = N \left[-\frac{c}{16\pi G} \left(\frac{3\dot{z}^2}{2N^2\sqrt{z}} + 6k\sqrt{z} - 2\Lambda z^{\frac{3}{2}} \right) - \frac{\dot{z}\dot{Q}}{N} - \frac{\dot{N}zQ}{N^2} + 18Bk\frac{\dot{z}^2}{N^2z^{\frac{3}{2}}} + \frac{N^2\sqrt{z}}{36B}Q^2 - \frac{36Bk^2}{\sqrt{z}} \right] \quad (39)$$

which is constrained to vanish. Now, using the expression,

$$p_Q p_z = \dot{z}\dot{Q} + \frac{3c\dot{z}^2}{16\pi GN\sqrt{z}} + \frac{\dot{N}}{N}\dot{z}Q - \frac{36Bk\dot{z}^2}{Nz^{\frac{3}{2}}}, \quad (40)$$

the Hamiltonian constraint equation in terms of the phase space variables is obtained as,

$$H_c = \frac{c}{16\pi G} \left(\frac{3p_Q^2}{2N\sqrt{z}} - 6Nk\sqrt{z} + 2N\Lambda z^{\frac{3}{2}} \right) - p_Q p_z - \frac{18Bk}{Nz^{\frac{3}{2}}}p_Q^2 + \frac{N^3\sqrt{z}}{36B}Q^2 - \frac{36BNk^2}{\sqrt{z}} = 0. \quad (41)$$

Finally, to express $H_c = N\mathcal{H}$, let us choose as before,

$$x = \frac{\dot{z}}{N}$$

$$Q = \frac{\partial A}{\partial \dot{z}} = \frac{\partial A}{\partial \dot{x}} \frac{d\dot{x}}{d\dot{z}} = \frac{p_x}{N}, \text{ and, } p_Q = -\dot{z} = -Nx,$$

So we need to replace Q by $\frac{p_x}{N}$ and p_Q by $-Nx$ in the above Hamiltonian. Thus we get,

$$H_c = N \left[xp_z + \frac{\sqrt{z}}{36B} p_x^2 - \frac{18Bk}{z^{\frac{3}{2}}} x^2 - 36B \frac{k^2}{\sqrt{z}} + \frac{c}{16\pi G} \left(\frac{3x^2}{2\sqrt{z}} - 6k\sqrt{z} + 2\Lambda z^{\frac{3}{2}} \right) \right] = N\mathcal{H} = 0. \quad (42)$$

It is important to notice that momentum is not associated with the Einstein-Hilbert sector of the action. The action (36) can now be expressed in the canonical form with respect to the basic variables as,

$$A = \int (\dot{z}p_z + \dot{x}p_x - N\mathcal{H}) dt \, d^3x = \int (\dot{h}_{ij}\pi^{ij} + \dot{K}_{ij}\Pi^{ij} - N\mathcal{H}) dt \, d^3x \quad (43)$$

The corresponding quantum version is

$$\frac{i\hbar}{\sqrt{z}} \frac{\partial \Psi}{\partial z} = -\frac{\hbar^2}{36Bx} \left(\frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi - 18Bk \left(\frac{x}{z^2} + \frac{2k}{zx} \right) \Psi + \frac{c}{16\pi G} \left(\frac{3x}{2z} - 6\frac{k}{x} + 2\Lambda \frac{z}{x} \right) \Psi. \quad (44)$$

Again under a further change of variable, the above equation takes the look of the Schrödinger equation, viz.,

$$i\hbar \frac{\partial \Psi}{\partial \alpha} = -\frac{\hbar^2}{54B} \left(\frac{1}{x} \frac{\partial^2}{\partial x^2} + \frac{n}{x^2} \frac{\partial}{\partial x} \right) \Psi + \left[\frac{c}{16\pi G} \left(\frac{x}{\alpha^{\frac{2}{3}}} - \frac{4k}{x} + \frac{4\Lambda\alpha^{\frac{2}{3}}}{3x} \right) - 12Bk \left(\frac{x}{\alpha^{\frac{4}{3}}} + \frac{2k}{x\alpha^{\frac{2}{3}}} \right) \right] \Psi = \hat{H}_e \Psi, \quad (45)$$

where, $\alpha = z^{\frac{2}{3}}$ again plays the role of internal time parameter and the effective potential V_e , given by,

$$V_e = \frac{c}{16\pi G} \left(\frac{x}{\alpha^{\frac{2}{3}}} - \frac{4k}{x} + \frac{4\Lambda\alpha^{\frac{2}{3}}}{3x} \right) - 12Bk \left(\frac{x}{\alpha^{\frac{4}{3}}} + \frac{2k}{x\alpha^{\frac{2}{3}}} \right), \quad (46)$$

is a function of both the so called time variable α and x , as before. We observe that Einstein-Hilbert sector appears as effective potential in the quantum version of the theory. We can try to explore the reason for this. In the case of E-H action alone, the momentum is $p_z \propto \dot{z}$. But here, under the introduction of auxiliary variable, one gets, $p_Q = -\dot{z}$, while a complicated expression for p_z , gives \dot{Q} . With the choice of basic variable, this p_Q has been transformed to a generalized co-ordinate x , which finally appears in the effective potential.

The effective Hamiltonian (\hat{H}_e) is again hermitian and so, one to write the continuity equation for $n = -1$ as before, and the probability interpretation follows.

4 Summary

Summarily, in an earlier work [15], we have presented a unique boundary term to supplement curvature squared action like the Gibbons-Hawking term, which supplements Einstein-Hilbert action. This boundary term is at par with the one obtained under the variation (metric formalism) of $F(R)$ theory of gravity, choosing $F(R) = R^2$. Such a boundary term usually can not be expressed as a total time derivative term. However, we have shown that, the boundary term $\Sigma = \sigma_1 + \sigma_2$, where, both are total derivative terms, under appropriate choice of auxiliary variable. In view of such a boundary term, here, we have been able to express the curvature squared action in the canonical form, in the presence of lapse function, taking Robertson-Walker minisuperspace model under consideration. The canonical form of the action is in close resemblance with the one for Einstein-Hilbert action. The phase space of the curvature squared action, being a higher derivative theory of gravity, is larger and is spanned by $h_{ij}, K_{ij}, \pi^{ij}, \Pi^{ij}$. Canonical quantization then leads to a Schödinger like equation, where, $\alpha = a^3$, the proper volume, acts as the internal time parameter. The effective Hamiltonian is self adjoint and so the probabilistic interpretation is straight forward. To express the continuity equation in the standard form, one has to fix up the operator ordering parameter $n = -1$. Since the effective Hamiltonian has time dependance through the effective potential, so the unitary transformation is given by the Dyson series. Finally, the extremization of the effective potential gives coasting solution for the curvature parameter $k = 1$, which implies that inflation is an artefact of curvature squared action for a initially closed Universe. Although, the whole process of canonizing has been carried out in a minisuperspace model, however, it might give insight to extend the work in the whole superspace.

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